

Some exact results on the QCD critical point

Yoshimasa Hidaka¹ and Naoki Yamamoto^{2,3,4}

¹Theoretical Research Division, Nishina Center, RIKEN, Wako 351-0198, Japan

²Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

³Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195-1550, USA

⁴Maryland Center for Fundamental Physics, Department of Physics, University of Maryland, College Park, MD 20742-4111, USA

E-mail: hidaka@riken.jp, nyama@umd.edu

Abstract. We show, in a model-independent manner, that the QCD critical point can appear only inside the pion condensation phase of the phase-quenched QCD as long as the contribution of flavor-disconnected diagrams is negligible. The sign problem is known to be maximally severe in this region, implying that the QCD critical point is reachable by the present lattice QCD techniques only if there is an enhancement of the flavor-disconnected contribution at finite baryon chemical potential.

1. Introduction and conclusion

The phase structure of quantum chromodynamics (QCD) at finite temperature T and finite baryon chemical potential μ_B has been a subject of considerable interest. Among others, of particular interest is the possible QCD critical point, which is the terminal point of the first-order chiral phase transition (for a review, see [1]). Such critical points may appear not only in hot matter between the hadron phase and quark-gluon plasma [2] but also in dense matter between the hadron phase and color superconductor [3, 4], and are potentially observable in heavy-ion collision experiments at RHIC and FAIR. Due to the sign problem, however, it has not yet been established in first principles lattice QCD simulations whether the QCD critical points really exist or not.

The first attempt to locate the QCD critical point was made by Fodor and Katz based on the reweighting method [5]. As pointed out by Splittorff [6], however, their critical points happen to be located on the critical line of the phase-quenched QCD with the fermion determinant $\det \mathcal{D}$ replaced by $|\det \mathcal{D}|$ (see Figure 1 for the phase diagram), where the reweighting method breaks down. See also [9, 10] for other but related problems. Indeed in every model calculation (at least in the mean-field approximation), such as the random matrix model [12], Nambu-Jona-Lasinio (NJL) model [13], and Polyakov-loop extended NJL model [14], critical points are somehow observed only inside the critical line of the phase-quenched theory where the sign problem is maximally severe [11, 6]. The question is whether this is the case in real QCD; if so, the QCD critical point is quite difficult to access by the present lattice QCD techniques.

In this article we show, in a model-independent manner, that the QCD critical point cannot exist under the following conditions [15]:

- (i) The contribution of flavor-disconnected diagrams is negligible compared with that of connected diagrams.

- (ii) The coordinate (T, μ) in QCD at finite $\mu_B = 3\mu$ is outside the critical line of the phase-quenched QCD. (Note that it is *not* a physical phase transition in QCD at finite μ_B .)

The condition (i) is satisfied in the large- N_c QCD and the models in the mean-field approximation, but it could be violated in real QCD. In other words, if the QCD critical point exists in a region accessible by the present lattice QCD techniques, the condition (i) must be broken; it necessitates some kind of enhancement of the flavor-disconnected contribution, which should also be observed in the lattice QCD simulations (see section 3 for more details).

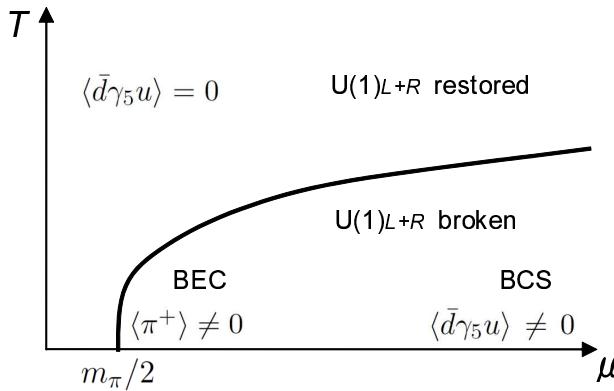


Figure 1. Phase diagram of phase-quenched QCD with μ or QCD with isospin chemical potential $\mu_I = 2\mu$ [7] (figure taken from [8]). See the text in section 2.3 for further detail.

2. Theorem on the QCD critical point

2.1. Sketch of the proof

The main observation to arrive at the conclusion above is that pions, which are the (pseudo-)Nambu-Goldstone modes associated with chiral symmetry breaking, are the lightest among all the mesonic modes (even in the medium) as long as the above conditions (i) and (ii) are met [15].¹ We empirically know that this is true in the QCD vacuum. If this is also true in some region at finite T and/or μ_B , the flavor-singlet mesonic mode σ , which has the quantum number $\bar{\psi}\psi$ with the mass m_σ , satisfies the relation:

$$m_\sigma \geq m_\pi > 0, \quad (1)$$

where m_π is the pion mass and the latter inequality follows due to the explicit breaking of chiral symmetry in the presence of the quark mass.

On the other hand, the definition of the QCD critical point is that the static correlation length ξ defined by

$$\langle \bar{\psi}\psi(\mathbf{x})\bar{\psi}\psi(\mathbf{y}) \rangle \sim e^{-|\mathbf{x}-\mathbf{y}|/\xi}, \quad (2)$$

is divergent, or $m_\sigma = 1/\xi$ is vanishing at the point. However, from the bound (1) for m_σ , this is apparently impossible, and thus, the QCD critical point cannot exist in this region. Below we shall show, based on the rigorous QCD inequalities [17], that pions are indeed the lightest mesonic modes under the conditions (i) and (ii) given in section 1. We shall work in the Euclidean and two-flavor QCD with the degenerate quark mass m . (For the generalization to QCD with any number of flavors N_f , see [15].)

¹ It is interesting to note that the maximal severity of the sign problem inside the critical line of the phase-quenched QCD is also related to the lightness of pions [16] (see also [11]).

2.2. Proof at $\mu_B = 0$

For clarity, we first illustrate the essence of the idea at $\mu_B = 0$, and will generalize it to the case for $\mu_B \neq 0$. Consider the Dirac operator $\mathcal{D} = D + m$, where the operator $D = \gamma_\mu(\partial_\mu + igA_\mu)$ satisfies the anti-Hermiticity and chiral symmetry, $D^\dagger = -D$ and $\gamma_5 D \gamma_5 = -D$. From these two properties, we can easily show

$$\gamma_5 \mathcal{D} \gamma_5 = \mathcal{D}^\dagger, \quad (3)$$

and the positivity, $\det \mathcal{D} \geq 0$.

Let us now consider a correlation function for a generic flavor nonsinglet fermion bilinear $M_\Gamma = \bar{\psi} \Gamma \psi$,

$$C_\Gamma(x, y) \equiv \langle M_\Gamma(x) M_\Gamma^\dagger(y) \rangle_{\psi, A} = -\langle \text{tr}[S_A(x, y) \Gamma S_A(y, x) \bar{\Gamma}] \rangle_A. \quad (4)$$

Here, $S_A(x, y) \equiv \langle x | \mathcal{D}^{-1} | y \rangle$ denotes a propagator from the point y to x in a background gauge field A , the symbols $\langle \cdot \rangle_{\psi, A}$ and $\langle \cdot \rangle_A$ denote the average over ψ, A and the average over A , respectively, and $\bar{\Gamma} \equiv \gamma_0 \Gamma^\dagger \gamma_0$. From (3) and the positivity, we have

$$C_\Gamma = \langle \text{tr}[S_A(x, y) \Gamma i \gamma_5 S_A^\dagger(x, y) i \gamma_5 \bar{\Gamma}] \rangle_A \leq \langle \text{tr}[S_A(x, y) S_A^\dagger(x, y)] \rangle_A. \quad (5)$$

Here we used the Cauchy-Schwarz inequality, which is saturated when $\Gamma = i \gamma_5 \tau_A$ with τ_A being the traceless flavor generators. The asymptotic behavior of C_Γ at large distance $|\mathbf{x} - \mathbf{y}|$ in the static limit can be written as

$$C_\Gamma \sim e^{-m_\Gamma |\mathbf{x} - \mathbf{y}|}, \quad (6)$$

where m_Γ is the mass of the lowest mesonic state in the channel Γ . Then the inequality (5) leads to

$$m_\Gamma \geq m_\pi. \quad (7)$$

In this derivation of the QCD inequalities, we need to assume that M_Γ is not flavor singlet, otherwise the contribution of flavor-disconnected diagrams $\langle \text{tr}[\Gamma S_A(x, x)] \text{tr}[\bar{\Gamma} S_A(y, y)] \rangle_A$ must also be taken into account. Phenomenologically disconnected diagrams may be suppressed compared with connected ones, as is known as the Okubo-Zweig-Iizuka (OZI) rule. Theoretically it can be justified in the 't Hooft large- N_c limit [18], since the flavor-disconnected diagrams are shown to be subleading in $1/N_c$ compared with the connected ones. Here we simply assume that the contribution of the flavor-disconnected diagrams is negligible [condition (i)], where (7) is also applicable to the flavor-singlet channel, $\Gamma = \sigma$.

This argument is valid for any temperature T . Therefore, we conclude that the QCD critical point cannot exist on the T -axis (at $\mu_B = 0$) under the condition (i).

2.3. Proof at $\mu_B \neq 0$

Unfortunately the above argument cannot be naively generalized to $\mu_B \neq 0$, because the Dirac operator $\mathcal{D}(\mu) = D + \mu \gamma_0 + m$ does no longer satisfy (3) and the positivity of the measure is lost. This is indeed the origin of the notorious sign problem in QCD at $\mu_B \neq 0$.

On the other hand, we can generalize it to the phase-quenched QCD, or equivalently, QCD at finite isospin chemical potential $\mu_I = 2\mu$. In this case, the Dirac operator $\mathcal{D}(\mu_I) = D + \mu_I \gamma_0 \tau_3/2 + m$ satisfies the relation $\tau_1 \gamma_5 \mathcal{D} \gamma_5 \tau_1 = \mathcal{D}^\dagger$ and $\det \mathcal{D}(\mu_I) \geq 0$ [7]; or complex Dirac eigenvalues for two flavors are conjugate with each other, and $\det \mathcal{D}(\mu_I) = |\det(\mu)| \geq 0$. Then the following inequality follows at *any* μ_I [7]:

$$C_\Gamma \leq \langle \text{tr}[S_A(x, y) S_A^\dagger(x, y)] \rangle_A, \quad (8)$$

which is saturated when $\Gamma = i\gamma_5\tau_{1,2}$. This leads to the inequality, $m_\Gamma \geq m_{\pi\pm}$. Under the condition (i) this is again applicable to $\Gamma = \sigma$, and repeating a similar argument to QCD at $\mu_B = 0$, the critical point would be prohibited at $\mu_I \neq 0$.

However, one should note that this is not necessarily applicable to any coordinate (T, μ_I) . Indeed when the pion condensation phase appears (where $m_\pm = 0$) as we will see shortly, $m_\sigma = 0$ is still possible and the QCD critical point is not ruled out.²

Let us look at the phase diagram of the phase-quenched QCD (figure 1), first discussed in [7]. As one increases μ_I at $T = 0$, π_+ meson is first excited (instead of a baryon) so that it exhibits the Bose-Einstein condensation (BEC) for $\mu > m_\pi/2$. On the other hand, at large μ_I where fundamental degrees of freedom are quarks and gluons, the attractive interaction between u and \bar{d} near the Fermi surface leads to the Bardeen-Cooper-Schrieffer (BCS) pairing $\langle \bar{d}\gamma_5 u \rangle$ [7]. Because of the same symmetry breaking patterns $U(1)_{L+R} \rightarrow \mathbb{Z}_2$, it is plausible that there is no phase transition between the two regimes. This is the so-called BEC-BCS crossover phenomena. Below we simply call it the pion condensation phase. Combining the arguments above altogether, it follows that the critical point cannot appear in QCD at $\mu_I \neq 0$ outside the pion condensation phase under the condition (i).

Now we can generalize this result to QCD at $\mu_B \neq 0$, taking into account the following property which holds under the conditions (i) and (ii):

$$\langle \bar{\psi}\psi \rangle_{\mu_B=3\mu} = \langle \bar{\psi}\psi \rangle_{\mu_I=2\mu}, \quad (9)$$

where the same (T, μ) is taken in both hand sides. This property can be understood as follows [19]: the contribution of u and d to the chiral condensate are decoupled from each other under the condition (i), and there is a charge conjugation symmetry which flips the sign of the chemical potential for u and d independently. As $(\mu_u, \mu_d) = (\mu, \pm\mu)$ correspond to the baryon and isospin chemical potentials respectively, the relation (9) follows. However, this is not true inside the pion condensation phase of the phase-quenched QCD where u and d are coupled in the ground state, and hence, the condition (ii) is necessary [8, 20]. Therefore the locations and the orders of the chiral transition must be the same between QCD at $\mu_B \neq 0$ and phase-quenched QCD outside the pion condensation phase of the latter. The exactness of the phase quenching can be actually shown for *any* observable to $O(N_f/N_c)$ [8, 20] using the large- N_c orbifold equivalence [21] between QCD at μ_B and QCD at μ_I (see also [22] for the orbifold equivalence between QCD and $SO(2N_c)$ gauge theory at $\mu_B \neq 0$).

Since the QCD critical point is already shown to be excluded outside the pion condensation phase of the phase-quenched QCD, the same must be true for QCD at $\mu_B \neq 0$. This is our main result [15]. This also provides the underlying reason why the critical points are always observed in this region with maximal severity of the sign problem in the model calculations [12, 13, 14].

3. Discussion

So far we have assumed the condition (i), but how is this reasonable in real QCD? To understand it, first note that as far as we assume the condition (i), the chiral transition cannot be first order outside the pion condensation phase from our theorem. This is because, if it was first order, we could weaken the phase transition by gradually increase the quark mass m and we would eventually arrive at the critical mass $m = m_c$ where the chiral transition becomes second order; this is prohibited from our theorem. It means that the only thing that can make the chiral transition stronger to be first order is the effect of flavor-disconnected diagrams, including the effect of the axial anomaly [23]. So whether the QCD critical point can appear outside the pion

² The statement in [15] that the critical point cannot appear in QCD at any μ_I in the large- N_c limit was not precise. However, the conclusion that the QCD critical point is ruled out at finite μ_B under the conditions (i) and (ii) remains unchanged.

condensation phase is determined by the competition between the contribution of disconnected diagrams and the quark mass. We know from the lattice QCD simulation [24] that the latter overwhelms the former at $\mu_B = 0$ to make the chiral transition crossover at finite T , and the condition (i) is satisfied in this sense. Therefore, unless there is enhancement of the latter such that it overcomes the former at $\mu_B \neq 0$ for some reason, the critical point would not be accessible by the present lattice QCD techniques.

In addition, nuclear matter is shown to lie inside this pion condensation phase independently of the quark mass, at least $T = 0$ [25]. Most likely these physics at finite baryon density appear due to the subtle cancellation of the complex phase. It is certainly an important challenge for the theory to find out other ways to sample appropriate configurations in this region, which could enable us to probe the interesting physics in dense QCD including the color superconductivity. (For such recent attempts, see, e.g., [26, 27, 16, 28].)

Acknowledgments

Y.H. was supported by JSPS KAKENHI Grant Numbers 23340067 and 24740184. N.Y. is supported by JSPS Research Fellowships for Young Scientists.

References

- [1] Stephanov M A 2004 *Prog. Theor. Phys. Suppl.* **153** 139 (*Preprint* hep-ph/0402115)
- [2] Asakawa M and Yazaki K 1989 *Nucl. Phys. A* **504** 668; Barducci A, Casalbuoni R, Curtis S De, Gatto R and Pettini G 1989 *Phys. Lett. B* **231** 463
- [3] Kitazawa M, Koide T, Kunihiro T and Nemoto Y 2002 *Prog. Theor. Phys.* **108** 929 (*Preprint* hep-ph/0207255)
- [4] Hatsuda T, Tachibana M, Yamamoto N and Baym G 2006 *Phys. Rev. Lett.* **97** 122001 (*Preprint* hep-ph/0605018); 2007 *Phys. Rev. D* **76** 074001 (*Preprint* 0704.2654); Abuki H, Baym G, Hatsuda T and Yamamoto N 2010 *Phys. Rev. D* **81** 125010 (*Preprint* 1003.0408)
- [5] Fodor Z and Katz S D 2002 *JHEP* **0203** 014 (*Preprint* hep-lat/0106002); 2004 *JHEP* **0404** 050 (*Preprint* hep-lat/0402006)
- [6] Splittorff K 2006 *PoS LAT* **2006** 023 (*Preprint* hep-lat/0610072)
- [7] Son D T and Stephanov M A 2001 *Phys. Rev. Lett.* **86** 592 (*Preprint* hep-ph/0005225)
- [8] Hanada M and Yamamoto N 2012 *JHEP* **1202** 138 (*Preprint* 1103.5480)
- [9] Ejiri S 2006 *Phys. Rev. D* **73** 054502 (*Preprint* hep-lat/0506023)
- [10] Golterman M, Shamir Y and Svetitsky B 2006 *Phys. Rev. D* **74** 071501 (*Preprint* hep-lat/0602026)
- [11] Splittorff K and Verbaarschot J J M 2007 *Phys. Rev. Lett.* **98** 031601 (*Preprint* hep-lat/0609076)
- [12] Han J and Stephanov M A 2008 *Phys. Rev. D* **78** 054507 (*Preprint* 0805.1939)
- [13] Andersen J O, Kyllingstad L T and Splittorff K 2010 *JHEP* **1001** 055 (*Preprint* 0909.2771)
- [14] Sakai Y, Sasaki T, Kouno H and Yahiro M 2010 *Phys. Rev. D* **82** 096007 (*Preprint* 1005.0993)
- [15] Hidaka Y and Yamamoto N 2012 *Phys. Rev. Lett.* **108** 121601 (*Preprint* 1110.3044)
- [16] Grabowska D, Kaplan D B and Nicholson A N *Preprint* 1208.5760
- [17] Weingarten D 1983 *Phys. Rev. Lett.* **51** 1830; Witten E 1983 *Phys. Rev. Lett.* **51** 2351; Nussinov S 1984 *Phys. Rev. Lett.* **52** 966; Espriu D, Gross M and Wheater J F 1984 *Phys. Lett. B* **146** 67
- [18] 't Hooft G 1974 *Nucl. Phys.* **B72** 461; Witten E 1979 *Nucl. Phys.* **B160** 57
- [19] Cohen T D 2004 *Phys. Rev. D* **70** 116009; (*Preprint* hep-ph/0410156); Toublan D 2005 *Phys. Lett. B* **621** 145 (*Preprint* hep-th/0501069)
- [20] Hanada M, Matsuo Y and Yamamoto N 2012 *Phys. Rev. D* **86** 074510 (*Preprint* 1205.1030)
- [21] Bershadsky M and Johansen A 1998 *Nucl. Phys.* **B536** 141 (*Preprint* hep-th/9803249); Schmaltz M 1999 *Phys. Rev. D* **59** 105018 (*Preprint* hep-th/9805218); Strassler M J *Preprint* hep-th/0104032; Kovtun P, Ünsal M and Yaffe L G 2005 *JHEP* **0507** 008 (*Preprint* hep-th/0411177)
- [22] Cherman A, Hanada M and Robles-Llana D 2011 *Phys. Rev. Lett.* **106** 091603 (*Preprint* 1009.1623); Cherman A and Tiburzi B C 2011 *JHEP* **1106** 034 (*Preprint* 1103.1639)
- [23] Pisarski R D and Wilczek F 1984 *Phys. Rev. D* **29** 338
- [24] Aoki Y, Endrodi G, Fodor Z, Katz S D and Szabo K K 2006 *Nature* **443** 675 (*Preprint* hep-lat/0611014)
- [25] Cohen T D 2003 *Phys. Rev. Lett.* **91** 032002 (*Preprint* hep-ph/0304024)
- [26] Aarts G 2009 *Phys. Rev. Lett.* **102** 131601 (*Preprint* 0810.2089)
- [27] Cristoforetti M *et al.* [AuroraScience Collaboration] 2012 *Phys. Rev. D* **86** 074506 (*Preprint* 1205.3996)
- [28] Ohnishi A, Ichihara T and Nakano T Z *Preprint* 1211.2282